J. Engg. Math. 17 (1983) 55–72

Radiation of short surface waves by oscillating submerged smooth cylinders

A. M. AYAD*

Department of Mathematics, Imperial College, London S.W. 7, England.

(Received May 10, 1982)

SUMMARY

An analysis is made of the generation of surface waves by the time-periodic oscillations of smooth submerged cylinders in the limit when the cylinder is at a large depth h below the free surface, and the frequency $\omega/(2\pi)$ of the oscillations is high. Specifically, the parameters of the problem are such that $\omega^2 d_2/g \gg 1$ and $h/d_1 \gg 1$ subject to the condition that $\omega^2 d_1 d_2/(gh) \ll 1$ where g is the acceleration due to gravity and d_1 and d_2 denote the maximum and minimum diameters of the body wave maker. The amplitude of the radiated waves if found to be exponentially small and to relate to certain critical points inside the cylinder.

1. Introduction and formulation

A submerged smooth cylinder has its axis horizontal and parallel to the free surface of a body of deep water and is making oscillations, of small amplitude and high frequency, in a direction making an angle θ_1 with the free surface (Fig. 1). In the two-dimensional case, let the smooth

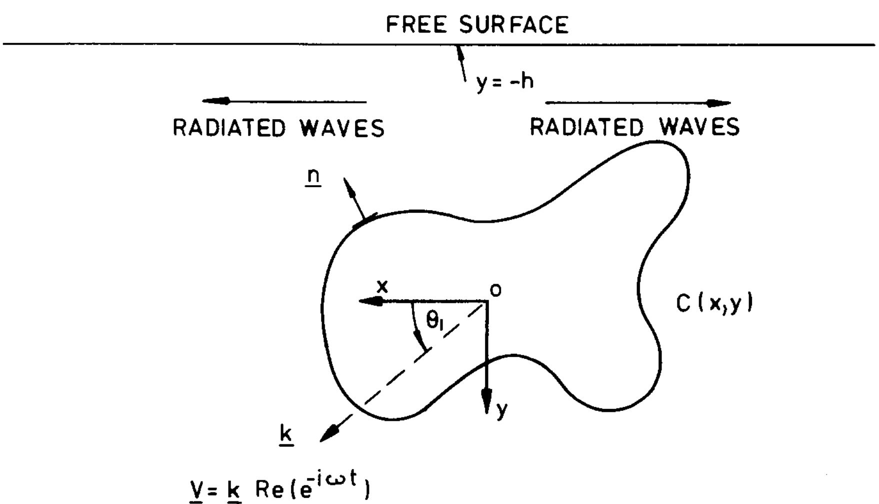

Figure 1. Cross-section of the oscillating cylinder.

*On leave from the University of Al-Fateh, Tripoli, Libya (S.P.L.A.J) cross-section of the cylinder be given by $C(x,y)$ and choose Cartesian coordinates (x,y) with origin at some convenient point inside $C(x,y)$, y pointing vertically downwards and x measured horizontally to the left. The free surface is at $y=-h$ with $h \gg d_1$, where d_1 is the maximum cylinder diameter. It is assumed that the fluid is inviscid and incompressible and that the motion is irrotational. For small time-periodic oscillations of the cylinder, the velocity potential is given by $\hat{\phi}\,(x, y, t) = \mathrm{Re}\{\hat{\phi}(x, y)e^{-i\omega t}\}$; $\omega/(2\pi)$ is the frequency of the oscillations. With the time factor suppressed, $\hat{\phi}(x,y)$ is specified by the following conditions:

$$\frac{\partial^2\hat{\phi}}{\partial x^2}+\frac{\partial^2\hat{\phi}}{\partial y^2} = 0 \qquad \text{in} \quad y > -h, \text{outside } C, \tag{1.1}$$

$$\hat{\phi}+\epsilon\frac{\partial\hat{\phi}}{\partial y} = 0 \qquad \text{on} \quad y = -h, \tag{1.2}$$

where $2\pi\epsilon = 2\pi g/\omega^2$ is the wavelength of the radiated waves and g is the acceleration due to gravity. It is assumed that $\epsilon \ll d_2$ and $\epsilon h/(d_1 d_2) \gg 1$, where d_2 is the minimum cylinder diameter. Also

$$\frac{\partial\hat{\phi}}{\partial n} = \mathbf{n}\cdot\mathbf{k} \qquad \text{on} \quad C, \tag{1.3}$$

where n is the unit outward normal to C and $\mathbf{k} = (\cos\theta_1, \sin\theta_1, 0)$ is the direction along which the wave maker is oscillating (Fig. 1). If the cylinder is oscillating with a constant velocity other than unity, then the linearity of the problem always permits its scaling out. The behaviour of $\hat{\phi}(x,y)$ at infinity is required to be that of outgoing waves only. Thus

$$\hat{\phi} \sim A\exp\left\{\frac{ix-y}{\epsilon}\right\} \qquad \text{as} \quad x \to +\infty, \tag{1.4}$$

$$\hat{\phi} \sim B\exp\left\{\frac{-ix-y}{\epsilon}\right\} \qquad \text{as} \quad x \to -\infty. \tag{1.5}$$

The constants A and B are unknowns of the problem and our aim is to estimate them when the body C is at a large distance below the free surface and the wavelength is small such that $\epsilon h/(d_1 d_2) \gg 1$. The next section deals with the case when $C(x, y)$ is a circle of unit radius, and the estimates obtained for A and B are thought to be valid in the limit when ϵ is very small, and h is very large compared with unity such that $\epsilon h \gg 1$.

In the third section we consider the case when C is an ellipse of arbitrary orientation. An estimate for B is obtained by utilising the approximations to the scattering potential obtained by Leppington [1]. The exponential behaviour of this estimate is found to relate to the focus nearest to the free surface.

An estimate for A can be arrived at along the same lines by simply reversing the direction of the incident wave in the scattering problem dealt with by Leppington [1] and using the corresponding approximations to the ensuing potential. Some corresponding results for arbitrary smooth cylinders are conjectured in the final section.

2. Radiation from a circular cylinder

If the cross-section of the oscillating cylinder is the unit circle $x^2 + y^2 = 1 = |z|^2, z = x + iy$, then the radiation potential $\hat{\phi}(x, y)$ is specified by conditions (1.1)–(1.5) with condition (1.3) taking the form

$$\frac{\partial \hat{\phi}}{\partial r} = \cos(\theta - \theta_1) = \tfrac{1}{2}\left\{ze^{-i\theta_1} + \frac{e^{i\theta_1}}{z}\right\} \qquad \text{on } |z| = 1, \tag{2.1}$$

where $z = e^{i\theta}$ on C and $\partial/\partial r$ indicates differentiation along the radius of the unit circle C.

When ϵ is very small and h is very large, it is plausible to suppose that a good approximation to $\hat{\phi}(x, y)$ is obtained by simply ignoring the free-surface condition (1.2) and solving for the potential $\hat{\phi}_0(x, y)$ that vanishes at infinity, is harmonic in $|z| > 1$, and satisfies condition (2.1) above. The potential $\hat{\phi}_0(x,y)$ is found with the aid of the Green's function

$$H(x,y;x_0,y_0) = \frac{1}{2\pi} \log \left\{ \frac{|z - z_0| \left| z - \dfrac{1}{\bar{z}_0} \right|}{|z|} \right\}, \tag{2.2}$$

where $\bar{z}_0$ is the complex conjugate of z_0 and $|z_0|^2 = x_0^2 + y_0^2 > 1$. In addition to being harmonic in $|z| > 1$ except for a source singularity at z_0, the function $H(x, y; x_0, y_0)$ has the property that:

$$\left.\begin{aligned} &\frac{\partial H}{\partial r} = 0 \\ &H = \frac{1}{2\pi} \log \left\{ |z_0| \left(1 - \frac{z}{z_0}\right) \left(1 - \frac{1}{z\bar{z}_0}\right) \right\} \end{aligned}\right\} \qquad \text{on } |z| = 1. \tag{2.3}$$

The application of Green's identity to $\hat{\phi}_0(x, y)$ and $H(x, y; x_0, y_0)$ in the domain $|z| > 1$, and the use of simple residue calculus yield the solution:

$$\hat{\phi}_0(x,y) = -\operatorname{Re}\left(\frac{e^{i\theta_1}}{z}\right) = -\frac{(x \cos\theta_1 + y \sin\theta_1)}{x^2 + y^2}. \tag{2.4}$$

This is not an exact solution of the problem posed originally since it fails to satisfy the free-surface condition (1.2). Following the procedure used by Leppington [1] in dealing with the problem, we define a harmonic correction potential $\hat{\phi}_c(x, y) = \hat{\phi}(x, y) - \hat{\phi}_0(x, y)$. Since

Since $\hat{\phi}_0$ is a wave-free potential, $\hat{\phi}_c(x, y)$ satisfies conditions (1.4) and (1.5) together with the specifications:

$$\frac{\partial \hat{\phi}_c}{\partial r} = 0 \qquad \text{on} \quad |z| = 1, \tag{2.5}$$

$$\hat{\phi}_c + \epsilon \frac{\partial \hat{\phi}_c}{\partial y} = q(x) \qquad \text{on} \quad y = -h, \tag{2.6}$$

where

$$q(x) = \frac{x \cos\theta_1 - h \sin\theta_1}{x^2 + h^2} - \epsilon \frac{\partial}{\partial x}\left(\frac{x \sin\theta_1 + h\cos\theta_1}{x^2 + h^2}\right). \tag{2.7}$$

Clearly, $\hat{\phi}_c(x, y)$ corresponds to the potential generated by a surface pressure distribution, proportional to $q(x)$, in the presence of a fixed circular cylinder. In the limit $\epsilon \to 0, h \to \infty$, the right hand side of (2.7) indicates that $q(x)$ is very small. Thus it is anticipated that $\hat{\phi}_0(x,y)$ will be good approximation to $\hat{\phi}(x,y)$ in this limit.

An exact solution for $\hat{\phi}_c(x, y)$ does not seem feasible. Instead it is supposed that, when h is very large, a potential which ignores the presence of the cylinder will be good approximation for $\hat{\phi}_c$, since the cylinder will appear to be a great distance away from the free surface. Abiding by this scheme, we write

$$\hat{\phi}_c(x,y) = \hat{\phi}_1(x,y) + \hat{\phi}_2(x,y), \tag{2.8}$$

where the harmonic function $\hat{\phi}_1(x, y)$ satisfies the radiation condition at infinity, of outgoing waves only, together with (2.6), and ignores condition (2.5), which has to be accounted for by $\hat{\phi}_2(x,y)$. An exact solution for $\hat{\phi}_1(x,y)$ follows from the known Green's function

$$G(x,y;x_1,y_1;\epsilon) = \frac{1}{4\pi} \log\left\{\frac{(x-x_1)^2 + (y-y_1)^2}{(x-x_1)^2 + (y+y_1+2h)^2}\right\}$$
$$-\frac{1}{\pi}\int_\Gamma \frac{e^{-\frac{t}{\epsilon}(y+y_1+2h)}}{t-1} \cos\left\{\frac{t}{\epsilon}(x-x_1)\right\} dt = G_0 + G_1 \text{ say.} \tag{2.9}$$

The integration path Γ runs from 0 to ∞, indented below the singularity at $t = 1$. The function $G(\mathrm{x};\mathrm{x}_1;\epsilon)$ is the potential due to a line source at (x_1, y_1) with no obstacles present.

The application of Green's identity, to $\hat{\phi}_1$ and G, in the region $y > -h$ yields:

$$\hat{\phi}_1(x_1,y_1) = \frac{1}{\epsilon}\int_{-\infty}^{\infty} G(x,-h;x_1,y_1)\, q(x)\, dx, \tag{2.10}$$

where $q(x)$ is given by (2.7).

In the integral (2.10), $G = G_1$ since the logarithmic term G_0 vanishes on $y = -h$. To evaluate $\hat{\phi}_1(x_1, y_1)$ we use the form

$$G_1(x, -h; x_1, y_1) = -\frac{1}{2\pi}\int_\Gamma \frac{e^{-\frac{t}{\epsilon}(h+y_1)}}{t-1}\{e^{\frac{it}{\epsilon}(x-x_1)} + e^{\frac{it}{\epsilon}(x_1-x)}\}\,dt \tag{2.11}$$

in (2.10). On interchanging the order of integration and performing the x-integral by a simple calculation of residues, it is found that:

$$\begin{aligned}\hat{\phi}_1(x,y) &= \frac{i}{2\epsilon}e^{-i\theta_1}\int_\Gamma\left(\frac{t+1}{t-1}\right)\exp\left\{-\frac{t}{\epsilon}(2h-iz)\right\}dt \\ &\quad -\frac{i}{2\epsilon}e^{i\theta_1}\int_\Gamma\left(\frac{t+1}{t-1}\right)\exp\left\{-\frac{t}{\epsilon}(2h+i\bar{z})\right\}dt\end{aligned} \tag{2.12}$$

where the point (x_1, y_1) is replaced by (x, y) with $z = x + iy$. If $x = \mathrm{Re}z \geqslant 0$ then, in (2.12), Γ is deformed onto the positive imaginary axis in the first integral and onto the negative imaginary axis in the second integral yielding:

$$\begin{aligned}\hat{\phi}_1(x,y) &= \frac{ie^{-i\theta_1}}{2\epsilon}\left[\frac{\epsilon}{2h-iz} + 4i\pi e^{-\frac{1}{\epsilon}(2h-iz)} + 2\int_0^\infty \frac{\exp\left\{-\frac{i\tau}{\epsilon}(2h-iz)\right\}d\tau}{\tau+i}\right] \\ &\quad -\frac{ie^{i\theta_1}}{2\epsilon}\left[\frac{\epsilon}{2h+i\bar{z}} + 2\int_0^\infty\frac{\exp\left\{\frac{i\tau}{\epsilon}(2h+i\bar{z})\right\}dt}{\tau-i}\right], \qquad \mathrm{Re}\,z \geqslant 0.\end{aligned} \tag{2.13}$$

Interchanging the deformations of Γ carried out in the case of $x \geqslant 0$, we arrive at:

$$\begin{aligned}\hat{\phi}_1(x,y) &= \frac{ie^{-i\theta_1}}{2\epsilon}\left[\frac{\epsilon}{2h-iz} + 2\int_0^\infty\frac{\exp\left\{\frac{i\tau}{\epsilon}(2h-iz)\right\}d\tau}{\tau-i}\right] \\ &\quad -\frac{ie^{i\theta_1}}{2\epsilon}\left[\frac{\epsilon}{2h+i\bar{z}} + 4i\pi e^{-\frac{1}{\epsilon}(2h+i\bar{z})} + 2\int_0^\infty\frac{\exp\left\{-\frac{i\tau}{\epsilon}(2h+i\bar{z})\right\}d\tau}{\tau+i}\right], \\ &\qquad \mathrm{Re}\,z \leqslant 0.\end{aligned} \tag{2.14}$$

It is now clear from (2.13) and (2.14) that

$$\hat{\phi}_1(x,y) \sim -\frac{2\pi}{\epsilon} e^{-i\theta_1 - \frac{2h}{\epsilon}} . e^{i\frac{x}{\epsilon} - \frac{y}{\epsilon}}, \qquad \text{as} \quad x \to +\infty,$$

$$\hat{\phi}_1(x,y) \sim \frac{2\pi}{\epsilon} e^{i\theta_1 - \frac{2h}{\epsilon}} . e^{-i\frac{x}{\epsilon} - \frac{y}{\epsilon}}, \qquad \text{as} \quad x \to -\infty. \tag{2.15}$$

Since, for very large h, $\hat{\phi}_1(x,y)$ is regarded as a good approximation to $\hat{\phi}_c(x,y)$ and since the latter has the specifications (1.4) and (1.5) at infinity, then it is concluded that

$$A \sim -\frac{2\pi}{\epsilon} e^{-i\theta_1 - \frac{2h}{\epsilon}} \qquad \text{for large } h, \tag{2.16}$$

and

$$B \sim \frac{2h}{\epsilon} e^{+i\theta_1 - \frac{2h}{\epsilon}} \qquad \text{for large } h. \tag{2.17}$$

The problem for $\hat{\phi}_2(x, y)$

In addition to the free-surface condition (1.2) and the condition

$$\frac{\partial \hat{\phi}_2}{\partial n} = -\frac{\partial \phi_1}{\partial n} \qquad \text{on} \quad |z| = 1, \tag{2.18}$$

the harmonic potential $\hat{\phi}_2(x,y)$, of (2.8), has the radiation condition of outgoing waves only at infinity. The problem for $\hat{\phi}_2(x,y)$ resembles the original one posed for $\hat{\phi}(x,y)$. The only difference between them is, of course, that the mode of oscillations of the circular cylinder is changed. In what follows, an attempt is made to show that, in the limit $\epsilon \to 0$, $h \to \infty$, the oscillation in (2.18) is smaller in absolute value than the original oscillation in (2.1). Differentiating (2.12) under the integral sign yields:

$$\left(\frac{\partial \hat{\phi}_1}{\partial n}\right)_{|z|=1} = -\frac{z e^{-i\theta_1}}{2\epsilon^2} \int_\Gamma \frac{t(t+1)}{t-1} \exp\left\{-\frac{t}{\epsilon}(2h - iz)\right\} dt$$
$$- \frac{\bar{z} e^{i\theta_1}}{2\epsilon^2} \int_\Gamma \frac{t(t+1)}{t-1} \exp\left\{-\frac{t}{\epsilon}(2h + i\bar{z})\right\} dt, \tag{2.19}$$

where $z\bar{z} = 1$ on C.

Deforming the contour Γ in a fashion similar to that which gave rise to the results (2.13) and (2.14) and integrating by parts we arrive at the result:

$$\left(\frac{\partial\hat{\phi}_1}{\partial n}\right)_{|z|=1} \sim \operatorname{Re}\left\{\frac{ze^{-i\theta_1}}{(2h-iz)^2}\right\} + 4\epsilon \operatorname{Re}\left\{\frac{ze^{-i\theta_1}}{(2h-iz)^3}\right\} \qquad \text{as} \quad \epsilon \to 0. \tag{2.20}$$

In the limit $\epsilon \to 0$, the dominant term in (2.20) will be the first one, and this in turn is small when h is very large. In fact

$$\left(\frac{\partial\hat{\phi}_1}{\partial n}\right)_{|z|=1} \sim \frac{1}{4h^2}\cos(\theta-\theta_1) \qquad \text{as} \quad \epsilon \to 0, \quad h \to \infty, \tag{2.21}$$

where $z = e^{i\theta}$ on C.

From (2.18) we have

$$\frac{\partial\hat{\phi}_2}{\partial n} \sim -\frac{1}{4h^2}\cos(\theta-\theta_1) \qquad \text{as} \quad \epsilon \to 0, \quad h \to \infty \text{ on } C. \tag{2.22}$$

Therefore, in the case of very high-frequency oscillations and when the centre of the cylinder is a large distance below the free surface, the oscillations in the problem for $\hat{\phi}_2$ is much smaller in absolute value than that in the original problem for $\hat{\phi}(x,y)$ {see (2.1)}. This leads to the belief that (2.16) and (2.17) are the leading terms in the expansions for A and B in the limit $\epsilon \to 0$, $h \to \infty$ such that $1/(\epsilon h) \to 0$. This is because the waves produced in the 'secondary' problem for $\hat{\phi}_2(x,y)$ will have much smaller amplitude in this limit. However, this argument is not rigorous since it appears to be heuristic in justifying that the neglected terms in the oscillations (2.22) make contributions, to A and B, asymptotically smaller than those in (2.16) and (2.17) in the limit under consideration. It may be appropriate here to point out that the difference between the swaying ($\theta_1 = 0$) and heaving ($\theta_1 = \pi/2$) cases is only a phase shift in the estimates for A and B. This is consistent with the results due to Ogilvie [2]. We conclude this section by pointing out that in the special case $\theta_1 = \pi/2$ (heaving) an approximation to the potential $\hat{\phi}(x,y)$ was found by, formally, setting $\epsilon = 0$ in condition (1.2) and using the conformal transformation

$$\zeta = \frac{az + ia^2}{az + i}, \qquad a = h - (h^2-1)^{1/2} < 1$$

to obtain a solution. By substituting this approximation in the formula obtained by applying Green's identity, in the fluid region, to the potential $\hat{\phi}(x,y)$ and the fundamental solution (2.9) due to a line source at (x_1, y_1), the same estimates obtained in (2.16) and (2.17) were arrived at (see Appendix).

3. Oscillations of an elliptic cylinder

In this section we investigate the case when the cross-section $C(x,y)$ of the wave maker is an ellipse inclined at an angle θ_0 to, and oscillating in a direction making an angle θ_1 with, the free surface $y = -h$ (Fig. 2).

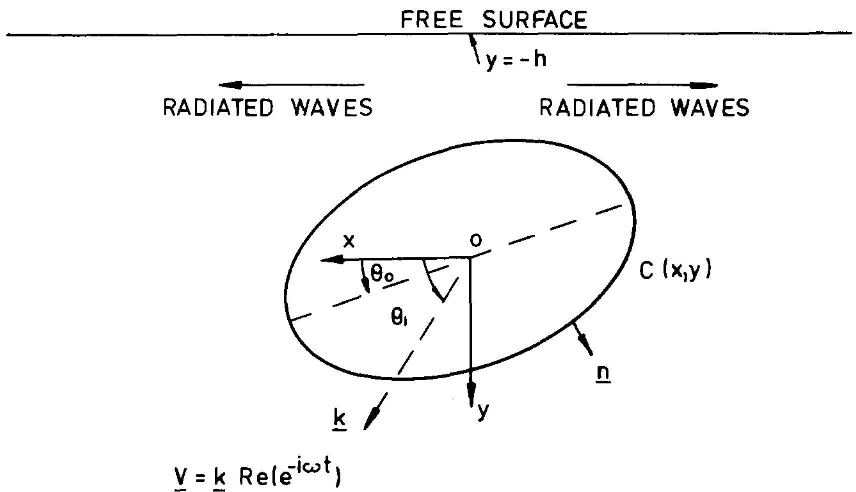

Figure 2. Oscillating elliptic cylinder.

Referred to the (x, y)-system of coordinates, the equation of the ellipse assumes the form:

$$\left(\frac{x\cos\theta_0 + y\sin\theta_0}{a}\right)^2 + \left(\frac{y\cos\theta_0 - x\sin\theta_0}{b}\right)^2 = 1, \tag{3.1}$$

with $a > b$.

The fluid motion, induced by the harmonic oscillations of the ellipse, has the velocity potential $\hat{\phi}(x, y)$ with the specifications (1.1)–(1.5) with (1.3), in this case, taking the form:

$$\frac{\partial\hat{\phi}}{\partial n} = \frac{\mu_0 x + \lambda_0 y}{(a^2 + b^2 - x^2 - y^2)^{1/2}} \quad \text{on} \quad C, \tag{3.2}$$

where

$$\begin{aligned} \mu_0 &= \frac{b}{a}\cos\theta_0\cos(\theta_1 - \theta_0) - \frac{a}{b}\sin\theta_0\sin(\theta_1 - \theta_0), \\ \lambda_0 &= \frac{b}{a}\sin\theta_0\cos(\theta_1 - \theta_0) + \frac{a}{b}\cos\theta_0\sin(\theta_1 - \theta_0). \end{aligned} \tag{3.3}$$

Our aim is to arrive at an estimate for the amplitude B of (1.5) in the short-wave limit ($\epsilon \ll b$) and when the centre of the ellipse is a large distance from the free surface ($h \gg a$). Towards the realisation of this, we make use of the approximations to the potential obtained by Leppington [1] when dealing with the corresponding scattering problem. This deals with the determination of the potential $\Phi(x, y)$ induced in the fluid by a plane wave $\exp\{(ix - y)/\epsilon\}$ incident from negative infinity in the presence of the same elliptic cylinder, which is now fixed. The potential

$\Phi(x,y)$ is specified by:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi = 0 \quad \text{in} \quad y > -h, \text{outside } C,$$

$$\frac{\partial \Phi}{\partial n} = 0 \quad \text{on} \quad C,$$

$$\Phi + \epsilon\frac{\partial \Phi}{\partial y} = 0 \quad \text{on} \quad y = -h, \tag{3.4}$$

$$\Phi \sim \begin{cases} Te^{\frac{ix-y}{\epsilon}} & \text{as} \quad x \to +\infty, \\ e^{\frac{ix-y}{\epsilon}} + Re^{\frac{-ix-y}{\epsilon}} & \text{as } x \to -\infty. \end{cases}$$

T and R are the transmission and reflection coefficients respectively. Green's theorem, applied to the radiation potential $\hat{\phi}(x, y)$ and $\Phi(x, y)$ of (3.4) in the fluid region, gives rise to the formula:

$$iBe^{\frac{2h}{\epsilon}} = \oint_C \Phi\frac{\partial \hat{\phi}}{\partial n}\,ds = \oint_C \left\{\frac{\mu_0 x + \lambda_0 y}{(a^2+b^2-x^2-y^2)^{1/2}}\right\}\Phi\,ds \tag{3.5}$$

where (3.2) is used and 's' denotes arc length. An exact solution to (3.4) does not seem to have been arrived at as yet, which means that B cannot be determined exactly. Therefore an estimate for B, which is thought to be valid in the limit $h/a \gg 1, b/\epsilon \gg 1$ such that $ab/(\epsilon h) \ll 1$, is sought instead.

Following Leppington [1] we write

$$\Phi = e^{\frac{iz}{\epsilon}} + \phi, \tag{3.6}$$

where $z = x + iy$ and ϕ is the scattered potential. Hence

$$iBe^{\frac{2h}{\epsilon}} = \oint_C \frac{(\mu_0 x + \lambda_0 y)e^{\frac{iz}{\epsilon}}\,ds}{(a^2+b^2-x^2-y^2)^{1/2}} + \oint_C \frac{(\mu_0 x + \lambda_0 y)\phi}{(a^2+b^2-x^2-y^2)^{1/2}} \tag{3.7}$$

The first integral in (3.7) can now be found using the conformal transformation

$$z = e^{i\theta_0}\left(\alpha\zeta + \frac{\beta}{\zeta}\right) \qquad \text{for} \qquad \left(\frac{\beta}{\alpha}\right)^{1/2} < |\zeta| < \infty, \tag{3.8}$$

where $\alpha = \frac{1}{2}\{a + b\}, \beta = \frac{1}{2}(a - b)$.

The ellipse C is then mapped onto the unit circle $|\zeta| = 1$. Hence

$$\oint_C \frac{(\mu_0 x + \lambda_0 y)e^{\frac{iz}{\epsilon}}\,ds}{(a^2+b^2-x^2-y^2)^{1/2}} = -i\oint_{|\zeta|=1}\left\{\bar{\mu}_1 + \frac{\mu_1}{\zeta^2}\right\}\exp\left\{\frac{ie^{i\theta_0}}{\epsilon}\left(\alpha\zeta+\frac{\beta}{\zeta}\right)\right\}d\zeta, \tag{3.9}$$

where $\mu_1 = \frac{1}{2}\{b\cos(\theta_1-\theta_0) + ia\sin(\theta_1-\theta_0)\}$.

But

$$\exp\left\{\frac{ie^{i\theta_0}}{\epsilon}\left(\alpha\zeta+\frac{\beta}{\zeta}\right)\right\} = \sum_{n=0}^{\infty} P_n\zeta^n + \sum_{n=1}^{\infty}\frac{Q_n}{\zeta^n} \quad \text{in} \quad \left(\frac{\beta}{\alpha}\right)^{1/2} < |\zeta| < \infty, \tag{3.10}$$

where

$$\begin{aligned} P_n &= i^n\left(\frac{\alpha}{\beta}\right)^{\frac{n}{2}} J_n\left(\frac{2}{\epsilon}e^{i\theta_0}\sqrt{\alpha\beta}\right), \\ Q_n &= i^n\left(\frac{\beta}{\alpha}\right)^{\frac{n}{2}} J_n\left(\frac{2}{\epsilon}e^{i\theta_0}\sqrt{\alpha\beta}\right) \end{aligned} \tag{3.11}$$

and $J_n(z)$ is the Bessel function of order n. Therefore

$$\oint_C \frac{(\mu_0 x + \lambda_0 y)e^{iz/\epsilon}\,ds}{(a^2+b^2-x^2-y^2)^{1/2}} = 2\pi(\mu_1 P_1 + \bar{\mu}_1 Q_1). \tag{3.12}$$

The scattered potential $\phi(x, y)$ in the second integral of (3.7) is not known exactly on the ellipse C. However, in the short-wave limit ($\epsilon \ll b$) and when $h \gg a$, an approximation $\phi_0(x,y)$ to $\phi(x, y)$ can be found in Leppington [1]. Briefly, the potential ϕ_0 is harmonic in the whole domain exterior to the ellipse C, vanishes at infinity and satisfies the condition

$$\frac{\partial\phi_0}{\partial n} = -\frac{\partial}{\partial n}(e^{iz/\epsilon}) \quad \text{on} \quad C. \tag{3.13}$$

The conformal transformation (3.8) and the Green's function (2.2), with z and z_0 replaced by ζ and ζ_0 respectively, lead to the image function ϕ_0^*. Thus

$$\phi_0^* = \sum_{n=1}^{\infty}\frac{P_n}{(\bar{\zeta})^n} - \sum_{n=1}^{\infty}\frac{Q_n}{\zeta^n} \quad \text{in} \quad 1 \leqslant |\zeta|, \tag{3.14}$$

where P_n and Q_n are as in (3.11). In particular

$$\phi_0^* = \sum_{n=1}^{\infty} P_n\zeta^n - \sum_{n=1}^{\infty}\frac{Q_n}{\zeta^n} \quad \text{on} \quad |\zeta| = 1. \tag{3.15}$$

The second integral in (3.7) can now be found in the limit under consideration. Thus

$$\oint_C \frac{(\mu_0 x + \lambda_0 y)\phi}{(a^2 + b^2 - x^2 - y)^{1/2}}\, ds \sim \oint_C \frac{(\mu_0 x + \lambda_0 y)\phi_0}{(a^2 + b^2 - x^2 - y^2)^{1/2}}\, ds$$

$$= -i \oint_{|\zeta|=1} \left(\bar{\mu}_1 + \frac{\mu_1}{\zeta^2}\right) \phi_0^*\, d\zeta = 2\pi(\mu_1 P_1 - \bar{\mu}_1 Q_1)$$

$$\text{as} \quad \frac{\epsilon}{b} \to 0, \qquad \frac{h}{a} \to \infty. \tag{3.16}$$

Results (3.12) and (3.16) give the estimate:

$$B \sim 2\pi \left(\frac{\alpha}{\beta}\right)^{1/2} \{b \cos(\theta_1 - \theta_0) + ia \sin(\theta_1 - \theta_0)\} e^{-(2h/\epsilon)} J_1\left(\frac{2e^{i\theta_0}}{\epsilon}\sqrt{\alpha\beta}\right)$$

$$\text{as} \quad \frac{\epsilon}{b} \to 0, \qquad \frac{h}{a} \to \infty. \tag{3.17}$$

It can be seen that (2.17) can be recovered from (3.17) by putting $a = b = 1$ (i.e. $\alpha = 1, \beta = 0$). This leads to the belief that (3.17) is valid in the limit $h/a \gg 1, b/\epsilon \gg 1$ such that $\epsilon h/(ab) \gg 1$.

Since all orientations of the ellipse can then be accounted for, θ_0 will be taken such that $0 \leqslant \theta_0 < \pi$. In this range, the asymptotics of the Bessel function give:

$$B \sim (2\pi\epsilon d)^{1/2} \left\{\frac{b \cos(\theta_1 - \theta_0) + ia \sin(\theta_1 - \theta_0)}{a - b}\right\} e^{-1/\epsilon(2h - d \sin\theta_0)}$$

$$\times \exp\left\{i\left(\frac{3\pi}{4} - \frac{\theta_0}{2} - \frac{d}{\epsilon}\cos\theta_0\right)\right\}$$

$$\text{as} \quad \frac{\epsilon}{b} \to 0, \quad \frac{h}{a} \to \infty \text{ such that } \frac{ab}{\epsilon h} \to 0, \tag{3.18}$$

where $d = 2(\alpha\beta)^{1/2}$ is the semi-focal distance.

It is seen from (3.18) that the exponential behaviour of the estimate for B depends crucially on the vertical depth $(h - d \sin\theta_0)$ of the focus nearest to the free surface. It may also be of some importance to note that this exponential behaviour does not depend on the angle of oscillation θ_1 which affects the multiplying factor only. In the case $\theta_0 = 0$, the major axis is horizontal and the two foci are equidistant from the free surface in which case the semi-focal distance d has a phase effect only.

We end this section by pointing out that a corresponding estimate for A of (1.4) can be obtained in the same manner by simply reversing the direction of the incident wave in problem (3.4) and using the corresponding approximations to the resulting scattered potential.

4. Generalisations

In this final section, it is our objective to generalise the method, outlined in Sec. 3, to oscillating cylinders of arbitrary smooth (continuously differentiable) and simple (no multiple points) cross-sections. To this end we start by considering the mapping:

$$z = \alpha_0 \zeta + \sum_{k=1}^{n} \frac{\alpha_k}{\zeta^k} \qquad \text{for} \quad 0 < \rho < |\zeta| < \infty \qquad (n \geqslant 1), \tag{4.1}$$

where $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ are, in general, complex constants and ρ is the modulus of the singularity {root of $dz/d\zeta = 0$} farthest from $\zeta = 0$. If we suppose that, under the map (4.1), the region exterior to $C(x, y)$ is transformed conformally onto the outside of C^* {$|\zeta| = 1$} with $C \to C^*$, then $\rho < 1$. This is because the closed curve C is, by hypothesis, simple and smooth and that no singularities can be allowed in the region outside C. The algebraic equation $dz/d\zeta = 0$ will, then, have roots on or within the closed disc $|\zeta| \leqslant \rho$ {Fig. 3} and the mapping (4.1) will, in fact, be conformal in the annulus $\rho < |\zeta| < \infty$. It is our intention, in what remains of this section, to keep to this branch of the mapping (4.1) bearing in mind that the images, of those singularities lying on $|\zeta| = \rho$, will be in the interior of $C(x, y)$ in the z-plane.

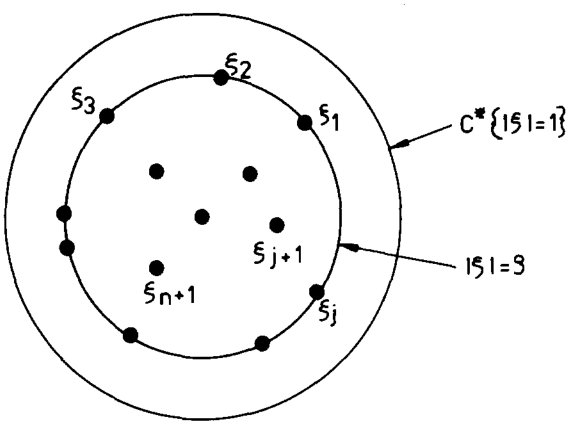

Figure 3. Critical points of the mapping.

The equation of the cross-section $C(x, y)$, of Fig. 1, can now be written down in the parametric form:

$$z = \alpha_0 e^{i\psi} + \sum_{k=1}^{n} \alpha_k e^{-ik\psi}, \tag{4.2}$$

where $\zeta = e^{i\psi}$ on C^*.

The condition on the body C, as given in (1.3), takes the form:

$$\frac{\partial\hat{\phi}}{\partial n} = \operatorname{Re}\left\{-ie^{-i\theta_1}\frac{dz}{ds}\right\} \qquad \text{on} \quad C, \tag{4.3}$$

where θ_1 is the angle of oscillation and s is the arc length. Using (4.2) and (4.3) we obtain:

$$\left\{\frac{\partial\hat{\phi}}{\partial n}\,ds\right\}_C = \left\{\bar{\nu}\zeta - \sum_{k=2}^{n}\bar{\nu}_k\zeta^k + \frac{\nu}{\zeta} - \sum_{k=2}^{n}\frac{\nu_k}{\zeta^k}\right\}\frac{d\zeta}{i\zeta} \qquad \text{on} \quad C^*, \tag{4.4}$$

where

$$\nu = \tfrac{1}{2}(\bar{\alpha}_0 e^{i\theta_1} - \alpha_1 e^{-i\theta_1}), \qquad \nu_k = \tfrac{1}{2}\,k\alpha_k e^{-i\theta_1}.$$

In order to arrive at an estimate for B of (1.5) we make use of the potential of problem (3.4) and Green's identity. Hence

$$iBe^{2h/\epsilon} = \oint_C \Phi\frac{\partial\hat{\phi}}{\partial n}\,ds. \tag{4.5}$$

Proceeding as in Sec. 3, we approximate $\Phi(x, y)$ by $\{e^{iz/\epsilon} + \phi_0\}$ with $\phi_0(x, y)$ having similar specifications as those outlined in that section when dealing with the elliptic cylinder case (Leppington [1]). Thus

$$iBe^{2h/\epsilon} \sim \oint_C \{e^{iz/\epsilon} + \phi_0\}\frac{\partial\hat{\phi}}{\partial n}\,ds \qquad \text{for} \qquad \frac{\epsilon}{d_2} \ll 1, \quad \frac{h}{d_1} \gg 1,$$

where d_1 and d_2 are the maximum and minimum diameters of $C(x, y)$ respectively.

Using the transformation (4.1) and formula (4.4), the integral can now be performed in the ζ-plane;

$$Be^{2h/\epsilon} \sim -\oint_{C^*}\left\{\phi_0^* + \exp\left[\frac{i}{\epsilon}\left(\alpha_0\zeta + \frac{\alpha_1}{\zeta} + \ldots + \frac{\alpha_n}{\zeta^n}\right)\right]\right\}\left\{\bar{\nu} - \sum_{k=2}^{n}\bar{\nu}_k\zeta^{k-1} + \frac{\nu}{\zeta^2} - \sum_{k=2}^{n}\frac{\nu_k}{\zeta^{k+1}}\right\}d\zeta \quad \text{for}\,\frac{d_2}{\epsilon} \gg 1, \frac{h}{d_1} \gg 1. \tag{4.7}$$

The exponential function can always be expanded in a Laurent series about $\zeta = 0$ valid in the region of analyticity of (4.1). Thus we let:

$$\exp\left\{\frac{i}{\epsilon}\left(\alpha_0\zeta + \frac{\alpha_1}{\zeta} + \ldots + \frac{\alpha_n}{\zeta^n}\right)\right\} = \sum_{m=0}^{\infty} R_m\zeta^m + \sum_{m=1}^{\infty}\frac{T_m}{\zeta^m} \quad \text{for} \quad \rho < |\zeta| < \infty, \tag{4.8}$$

where ρ is as in (4.1). The coefficients R_m and T_m are given by:

$$
\begin{aligned}
R_m &= \frac{1}{2i\pi}\oint_{\Gamma} \frac{1}{\zeta^{m+1}} \exp\left\{\frac{i}{\epsilon}\left(\alpha_0\zeta + \frac{\alpha_1}{\zeta} + \ldots + \frac{\alpha_n}{\zeta^n}\right)\right\} \mathrm{d}\zeta, \\
T_m &= \frac{1}{2i\pi}\oint_{\Gamma} \zeta^{m-1} \exp\left\{\frac{i}{\epsilon}\left(\alpha_0\zeta + \frac{\alpha_1}{\zeta} + \ldots + \frac{\alpha_n}{\zeta^n}\right)\right\} \mathrm{d}\zeta.
\end{aligned}
\tag{4.9}
$$

Γ is any circle with centre at $\zeta = 0$ and radius greater than ρ. The function ϕ_0^* can now be determined in the same way outlined prior to (3.14). Thus

$$
\phi_0^* = \sum_{m=1}^{\infty} \frac{R_m}{(\bar{\zeta})^m} - \sum_{m=1}^{\infty} \frac{T_m}{\zeta^m} \qquad \text{for} \quad |\zeta| \geqslant 1. \tag{4.10}
$$

Therefore

$$
\begin{aligned}
Be^{2h/\epsilon} &\sim -\oint_{|\zeta|=1} \left\{R_0 + 2\sum_{m=1}^{\infty} R_m\zeta^m\right\} \left\{\bar{\nu} - \sum_{k=2}^{n} \bar{\nu}_k\zeta^{k-1} + \frac{\nu}{\zeta^2} - \sum_{k=2}^{n} \frac{\nu_k}{\zeta^{k+1}}\right\} \mathrm{d}\zeta \\
&= 4i\pi\left\{-\nu R_1 + \sum_{k=2}^{n} \nu_k R_k\right\} \qquad \text{for} \quad \frac{d_2}{\epsilon} \gg 1, \quad \frac{h}{d_1} \gg 1.
\end{aligned}
\tag{4.11}
$$

It is thought that, in order for (4.11) to be the leading term, the ratio $d_1 d_2/(\epsilon h)$ should be small in the above limit. Result (3.17) can easily be recovered from (4.11) by letting $\alpha_0 = \alpha e^{i\theta_0}$, $\alpha_1 = \beta e^{i\theta_0}$ $(\alpha > \beta > 0)$ and $\alpha_k = 0 (k \geqslant 2)$.

In the limit $d_2/\epsilon \gg 1$, the largest contribution to the asymptotic behaviour of $R_k (k = 1, 2, \ldots, n)$ comes from the roots of $\mathrm{d}z/\mathrm{d}\zeta = 0$ (i.e. the singularities of the mapping (4.1)) as can be seen from the first of the integrals (4.9) (Copson [3]). In fact the only singularities which contribute to R_k are these which lie on the boundary of the region of analyticity of the branch function used in the mapping (i.e. on $|\zeta| = \rho$, see Fig. 3). The images of these singularities are inside $C(x, y)$ in the z-plane. It is suggested here that the dominant contribution to R_k will be due to that singularity with an image nearest to the free surface in the z-plane.

Conclusion

In the short-wave limit $(\epsilon/d_2 \ll 1)$ and when the oscillating cylinder $C(x, y)$ is at a large depth below the free surface $(d_1/h \ll 1)$ such that $d_1 d_2/(\epsilon h) \ll 1$, the cases considered in the preceding sections seem to indicate that the exponentially small amplitude always relate to that singularity of the mapping which lies inside C and on the boundary of the region of analyticity of the branch function used to transform the exterior of C onto the outside of the unit circle, and which has the smallest imaginary part (i.e. lying closest to the free surface). Singularities which lie at an equal depth below the free surface make equally important contributions to the asymptotic behaviour of the amplitude and their sum is taken in such a case. The method outlined in Sections 3 and 4 can be used to investigate the case when the points of $C(x,y)$ are oscillating with different velocities, i.e.

$$\mathrm{V} = \mathrm{Re}\{\mathrm{v}(x,y)e^{-i\omega t}\}.$$

Acknowledgements

I would like to express my gratitude to the authorities in the University of Al-Fateh, Tripoli, Libya (S.P.L.A.J.), for their financial support throughout my leave. I am also highly indebted to Dr. F. G. Leppington for all the valuable discussions and helpful suggestions during the course of this work.

Appendix: An alternative method for a heaving circular cylinder:

When $\theta_1 = \pi/2$ in condition (2.1) and ϵ is put equal to zero in the free-surface condition (1.2), an approximation $\phi_0(x, y)$ to the radiation potential $\hat{\phi}(x, y)$ which is expected to be reasonable in the short-wave limit is formulated as follows:

$$\begin{aligned}
&\frac{\partial^2\phi_0}{\partial x^2} + \frac{\partial^2\phi_0}{\partial y^2} = 0 \qquad \text{in} \quad y > -h, \quad \text{outside } |z| = 1, \\
&\phi_0 = 0 \qquad \text{on} \quad z = x - ih, \\
&\frac{\partial\phi_0}{\partial n} = \frac{i}{2}\left(\frac{1}{z} - z\right) \qquad \text{on} \quad |z| = 1, \\
&\phi_0 \to 0 \qquad \text{as} \quad |z| \to \infty.
\end{aligned} \tag{A.1}$$

In order to obtain the solution to (A. 1), the conformal mapping

$$\zeta = \mathrm{R}e^{i\psi} = \frac{az + ia^2}{az + i}, \qquad a = h - (h^2 - 1)^{1/2} < 1 \tag{A.2}$$

is used to deal with the image function ϕ_0^* in the ζ-plane. Bearing in mind that

$$\left\{\frac{\partial\phi_0}{\partial n}\right\}_{|z|=1} \to \left\{\frac{\partial\phi_0^*}{\partial R}\left|\frac{d\zeta}{dz}\right|\right\}_{|\zeta|=a},$$

the problem for ϕ_0^* becomes:

$$\nabla^2 \phi_0^* = 0 \quad \text{in} \quad a < |\zeta| < 1,$$
$$\phi_0^* = 0 \quad \text{on} \quad |\zeta| = 1, \tag{A.3}$$
$$\frac{\partial \phi_0^*}{\partial R} = f(\psi) = \frac{(1-a^2)(\cos\psi - \sigma)}{a(1+a^2)(1-\sigma\cos\psi)^2} \quad \text{on} \quad \zeta = ae^{i\psi},$$

where '∇^2' is the Laplacian operator and $\sigma = 2a/(1+a^2) < 1$. The last condition in (A.1) is automatically satisfied since $\zeta \to 1$ as $|z| \to \infty$ and $\phi_0^* = 0$ on $|\zeta| = 1$.

The Fourier expansion

$$f(\psi) = \frac{(1-a^2)}{a^2} \sum_{n=1}^{\infty} na^n \cos(n\psi), \qquad 0 \leqslant \psi < 2\pi, \tag{A.4}$$

and the method of separation of the variables R and ψ yield the solution

$$\phi_0^* = (1-a^2) \sum_{n=1}^{\infty} \frac{a^{2n-1}}{1+a^{2n}} (R^n - R^{-n}) \cos(n\psi), \qquad a \leqslant R \leqslant 1, \quad 0 \leqslant \psi \leqslant 2\pi. \tag{A.5}$$

The application of Green's theorem to the radiation potential $\hat{\phi}(x,y)$ of Sec. 2, with $\theta_1 = \pi/2$ in (2.1), and the Green's function (2.9), in the fluid region, leads to the formula:

$$\hat{\phi}(x_1, y_1) = \oint_{|z|=1} G \frac{\partial \hat{\phi}}{\partial n}\, ds - \oint_{|z|=1} \frac{\partial G}{\partial n}\, ds. \tag{A.6}$$

It can be shown that the wave-part of the Green's function is given by:

$$G_w = -i \exp\left\{\frac{i|x - x_1| - (y + y_1 + 2h)}{\epsilon}\right\}. \tag{A.7}$$

The wave-part of the radiation potential $\hat{\phi}(x_1, y_1)$ is, therefore, given by:

$$\hat{\phi}_w(x_1, y_1) = \oint_{|z|=1} G_w \frac{\partial \hat{\phi}}{\partial n}\, ds - \oint_{|z|=1} \hat{\phi} \frac{\partial G_w}{\partial n}\, ds. \tag{A.8}$$

Hence

$$\hat{\phi}_w(x_1, y_1) \sim -ie^{-(2h/\epsilon)} \left\{\oint_{|z|=1} e^{-(i/\epsilon z)} \frac{\partial \hat{\phi}}{\partial n}\, ds - \oint_{|z|=1} \hat{\phi} \frac{\partial}{\partial n}(e^{-i\bar{z}/\epsilon})\, ds\right\} e^{(ix_1 - y_1)/\epsilon}$$
$$\text{as} \quad x_1 \to +\infty. \tag{A.9}$$

Comparing (A.9) with (1.4) we see that:

$$A = -ie^{-(2h/\epsilon)}\left\{\oint_{|z|=1} e^{-(i/\epsilon z)}\frac{\partial\hat{\phi}}{\partial n}\,ds - \oint_{|z|=1}\hat{\phi}\frac{\partial}{\partial n}(e^{-i\bar{z}/\epsilon})\,ds\right\} \tag{A.10}$$

The first of the integrals in (A.10) can be easily evaluated by using condition (2.1) with $\theta_1 = \pi/2$. Hence

$$A = \frac{i\pi}{\epsilon}e^{-(2h/\epsilon)} + ie^{-(2h/\epsilon)}\oint_{|z|=1}\hat{\phi}\frac{\partial}{\partial n}(e^{-i\bar{z}/\epsilon})\,ds. \tag{A.11}$$

If in the limit $\epsilon \to 0$, $\phi_0(x, y)$ of (A.1) is regarded as a reasonable approximation to $\hat{\phi}(x, y)$, then

$$A \sim \frac{i\pi}{\epsilon}e^{-(2h/\epsilon)} + e^{-(2h/\epsilon)}\oint_{|z|=1}\phi_0\frac{\partial}{\partial n}(e^{-i\bar{z}/\epsilon})\frac{dz}{z} \qquad \text{as} \quad \epsilon \to 0. \tag{A.12}$$

Our knowledge of ϕ_0^* of (A.3) and (A.5) makes it easier to evaluate the integral of (A.12) in the ζ=plane, where ζ is given by (A.2). Thus

$$\oint_{|z|=1}\phi_0\frac{\partial}{\partial n}(e^{-i\bar{z}/\epsilon})\frac{dz}{z} =$$

$$\frac{(1-a^2)^2e^{a/\epsilon}}{2\epsilon}\sum_{n=1}^{\infty}\frac{(1-a^{2n})}{1+a^{2n}}\oint_{|\zeta|=a}\left(\zeta^n + \frac{a^{2n}}{\zeta^n}\right)e^{-a(1-a^2)/\epsilon(\zeta-a^2)}\frac{d\zeta}{(\zeta-a^2)^2}. \tag{A.13}$$

By expanding the function

$$F(\zeta) = \frac{1}{(\zeta-a^2)^2}\exp\left\{\frac{-a(1-a^2)}{\epsilon(\zeta-a^2)}\right\}$$

in Laurent series for $|\zeta| > a^2$ we arrive at the result:

$$\oint_{|z|=1}\phi_0\frac{\partial}{\partial n}(e^{-i\bar{z}/\epsilon})\frac{dz}{z} =$$

$$\frac{i\pi}{\epsilon} + \frac{2i\pi}{\epsilon}(1-a^2)^2e^{a/\epsilon}\sum_{k=1}^{\infty}\frac{(-1)^k a^{2k}}{(1-a^{2k+2})^2}\exp\left\{\frac{-(1-a^2)a^{2k+1}}{\epsilon(1-a^{2k+2})}\right\}. \tag{A.14}$$

In the limit $\epsilon \to 0, h \to \infty (a \to 0)$ such that $1/(\epsilon h) \to 0$ $(a/\epsilon \to 0)$, the contribution from the series is small compared to $(i\pi/\epsilon)$ and therefore (A.12) and (A.14) yield the estimate

$$A \sim \frac{2i\pi}{\epsilon} e^{-(2h/\epsilon)} \quad \text{as} \quad \epsilon \to 0, \quad h \to \infty \qquad \text{such that} \qquad \frac{1}{\epsilon h} \to 0. \tag{A.15}$$

This is the same as (2.16) with $\theta_1 = \pi/2$. The estimate (2.17) can be recovered in the same way by letting $x_1 \to -\infty$ in (A.7) and (A.8).

REFERENCES

[1] F. G. Leppington and P. F. Siew, Scattering of surface waves by submerged cylinders, *Appl. Ocean Research* 2 (1980) 129–137.

[2] T. F. Ogilvie, First- and second-order forces on a cylinder submerged under a free surface, *J. Fluid Mech.* 16 (1963) 451–472.

[3] E. T. Copson, *Asymptotic expansions*, Cambridge University Press, London (1965).

[4] G. N. Watson, *Theory of Bessel functions*, Second Edition, Cambridge University Press, London (1952).

[5] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Fourth Edition, Academic Press, New York, London (1965).